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Note

On a Conjecture of B. Grünbaum

IVAN HAVEL

Mathematical Institute, Czechoslovak Academy of Sciences, Prague 1, Czechoslovakia

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In this note we present a planar graph G with four 3-circuits which is not 3-colorable. No two of the 3-circuits of G have a vertex in common. This gives a negative answer to a conjecture formulated in [1].

Grünbaum proved [1] the following generalization of Grötzsch's theorem [2]:

Every planar graph with not more than three 3*-circuits is* 3*-colorable.*

The following conjecture appears in [1]:

If a planar graph G is not 3*-colorable then G contains two pairs of* (*edge or vertex*) *incident triangles.*

We present below (Proposition 2) a counterexample to this conjecture.

PROPOSITION 1. *Let G be the graph of Figure* 1 (*G contains two triangles which are not incident*). *Any* 3*-coloring of G assigns the same color to the vertices u and v.*

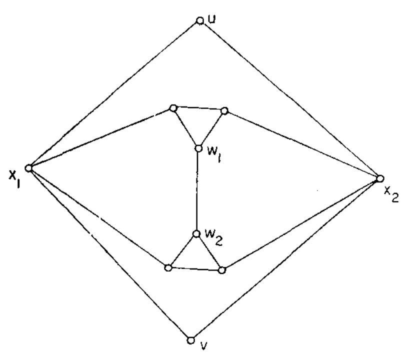

FIGURE 1

PROOF: Suppose on the contrary that u and v obtain different colors, say color 1 and color 2. Both x_1 and x_2 must then be colored by 3, hence w_1 and w_2 must also be colored by 3, which is impossible.

PROPOSITION 2. *Let G be the graph of Figure 2 (G contains four triangles, no two of them being incident). G is not 3-colorable.*

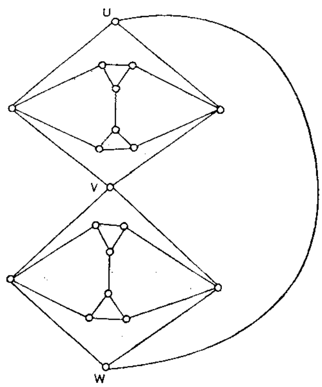

FIGURE 2

PROOF: Suppose, on the contrary, a 3-coloring of G be given. It follows from Proposition 1 that the vertices u and v are of the same color. Applying Proposition 1 once more we obtain that v and w are of the same color as well. Hence u and w are of the same color, which is impossible.

The construction was simplified by Grünbaum, who suggested a graph (shown in Figure 3) having fewer vertices than that of Figure 2. He also showed that the conjecture cannot even be saved by asking for 3-connectivity of the graph. (The counterexample is given in Figure 4.)

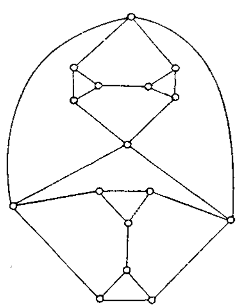

FIGURE 3

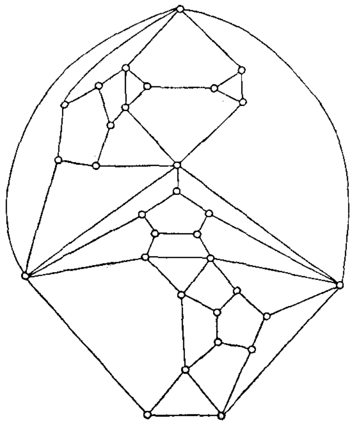

FIGURE 4

It seems that "distance" of the 3-circuits in a planar graph G plays an important role in the 3-colorability of G.

Let us define the distance of the 3-circuits in a graph G as the least integer n such that there is a path of length n in G joining two vertices of two different 3-circuits. The distance of (edge or vertex) incident 3-circuits is then 0.

We conclude with an open problem we were unable to solve: Is there an n_0 such that if the distance of the 3-circuits in a planar graph G is at least n_0 then G is 3-colorable? Is, possibly, $n_0 = 2$?

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2. H. GRÖTZSCH, Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe* **8** (1958), 109–120.